$(q;l,\lambda)$-deformed Heisenberg algebra: coherent states, their statistics and geometry

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Abstract

The Heisenberg algebra is deformed with the set of parameters $\{q,l,\lambda\}$ to generate a new family of generalized coherent states respecting the Klauder criteria. In this framework, the matrix elements of relevant operators are exactly computed. Then, a proof on the sub-Poissonian character of the statistics of the main deformed states is provided. This property is used to determine the induced generalized metric.

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1 Introduction

The Heisenberg algebra is generated by the the identity operator $\mathbf{1}$ and two mutually adjoint operators, b and its Hermitian conjugate $b^\dagger$ (also called annihilation and creation operators in Physics literature), satisfying the commutation relations

$$[b,\ b^\dagger] = \mathbf{1}, \qquad [b,\ \mathbf{1}] = 0 = [b^\dagger,\ \mathbf{1}], \tag{1.1}$$

where $[A,\ B] = AB - BA$. Defining the operator $N := b^\dagger b$, known as the *number operator*, the commutation relations (1.1) induce the two following properties:

$$[N,\ b] = -b \quad \text{and} \quad [N,\ b^\dagger] = b^\dagger. \tag{1.2}$$

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Let $\mathcal{F}$ be a Fock space and $\{|n\rangle \mid n \in \mathbb{N} \bigcup \{0\}\}$ be its orthonormal basis. The actions of b, $b^\dagger$ and N on $\mathcal{F}$ are given by

$$b|n\rangle = \sqrt{n}|n-1\rangle, \ \ b^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \ \text{ and } \ N|n\rangle = n|n\rangle \tag{1.3}$$

where $|0\rangle$ is a normalized vacuum:

$$b|0\rangle = 0, \qquad \langle 0|0\rangle = 1. \tag{1.4}$$

From (1.3) the states $|n\rangle$ for $n \geq 1$ are built as follows:

$$|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle, \ \ n = 1, \ 2, \ \cdots \tag{1.5}$$

satisfying the orthogonality and completeness conditions:

$$\langle m|n\rangle = \delta_{m,n}, \qquad \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1}. \tag{1.6}$$

Definition 1.1. The normalized states $|z\rangle \in \mathcal{F}$ for $z \in \mathbb{C}$ satisfying one of the following three equivalent conditions:

(i)

$$b|z\rangle = z|z\rangle, \qquad \langle z|z\rangle = 1 \tag{1.7}$$

or

(ii)

$$(\Delta Q)(\Delta P) = \frac{\hbar}{2} \tag{1.8}$$

where $(\Delta X)^2 := \langle z|X^2 - \langle X\rangle^2|z\rangle$ with $\langle X\rangle := \langle z|X|z\rangle$,

$$Q := (\hbar/2\mathbf{m}\omega)^{1/2}\,(b + b^\dagger), \qquad P := -i\,(\mathbf{m}\hbar\omega/2)^{1/2}\,(b - b^\dagger)$$

or

(iii)

$$|z\rangle = e^{zb^\dagger - \bar{z}b}|0\rangle \tag{1.9}$$

are called the coherent states (CS).

In the condition (ii), $\mathbf{m}$ stands for the particle mass; ω is the angular frequency. Explicitly, the canonical CS are computed as follows:

$$|z\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle = e^{-|z|^2/2}e^{zb^\dagger}|0\rangle, \ z \in \mathbb{C}. \tag{1.10}$$

In (1.10) and (1.9) we use the famous elementary Baker-Campbell-Hausdorff formula

$$e^{A+B} = e^{-\frac{1}{2}[A,B]}e^{A}e^{B} \tag{1.11}$$

whenever $[A,[A,B]] = [B,[A,B]] = 0$. The important feature of these coherent states resides in the partition (resolution) of unity:

$$\int_{\mathbb{C}} \frac{[d^2z]}{\pi}|z\rangle\langle z| = \sum_{n=0}^{\infty}|n\rangle\langle n| = \mathbf{1}, \tag{1.12}$$

where we have put $[d^2z] = d(Rez)d(Imz)$ for simplicity.

Definition 1.2. The unitary operator

$$D(z) := e^{zb^{\dagger}-\bar{z}b}, \qquad z \in \mathbb{C} \tag{1.13}$$

is called a coherent (displacement) operator.

From the property

$$D(z+w) = e^{-\frac{1}{2}(z\bar{w}-\bar{z}w)}D(w)D(z), \qquad z, \quad w \in \mathbb{C} \tag{1.14}$$

we infer the well-known commutation relation

$$D(z)D(w) = e^{(z\bar{w}-\bar{z}w)}D(w)D(z). \tag{1.15}$$

Coherent states were invented by Schrödinger in 1926 in the context of the quantum harmonic oscillator. They were defined as minimum-uncertainty states that exhibit its classical behavior [47]. In 1963, they have been simultaneous rediscovered by Glauber [19, 20], Klauder [27, 28] and Sudarshan [48] in quantum optics of coherent light beams emitted by lasers. Since there, they became very popular objects in mathematics (specially in functional analysis, group theory and representations, geometric quantization, etc.), and in nearly all branches of quantum physics (nuclear, atomic and solid state physics, statistical mechanics, quantum electrodynamics, path integral, quantum field theory, etc.). For more information we refer the reader to the references [2, 30, 41, 50].

The vast field covered by coherent states motivated their generalizations to other families of states deducible from noncanonical operators and satisfying not necessarily all above mentioned properties.

The first class of generalizations, based on the equivalent conditions given in Definition 1.1, include:

a) The approach by Barut and Girardello [8] considering coherent states as eigenstates of the annihilation operator. This approach was unsuccessful because of its drawbacks from both mathematical and physics point of view as detailed in [18, 41].

b) The approach based on the minimum-uncertainty states, i.e. essentially on the original motivation of Schrödinger in his construction of wavepackets which follow the motion of a classical particle while retaining their shapes. This was the basis for building the intelligent coherent states for various dynamical systems [4, 5, 38, 39, 40]. Nevertheless, as has been emphasized by Zhang *et al* [50], such a generalization has several limitations.

c) The approach related to the unitary representation of the group generated by the creation and annihilation operators. In two papers by Klauder [27, 28] devoted to a set of continuous states, one finds the basic ideas of coherent states construction for arbitrary Lie groups, which have been exploited by Gilmore [17] and Perelomov [41, 42] to formulate a general and complete formalism of building coherent states for various deformations of the Heisenberg group with properties similar to those of the harmonic oscillator. The key result of this development was the intimate connection of the coherent states with the dynamical group of a given physical system.

Two other generalizations complete this first class of generalizations: (*i*) the covariant coherent states introduced in Ref. [2], considered as a generalization of Gilmore-Perelomov formalism in the sense that the CS are built from more general groups (homogeneous spaces), and (*ii*) the nonlinear coherent states related to nonlinear algebras. Even though nonlinear coherent states have been used to analyze some quantum mechanical systems as the motion of a trapped ion [24, 36], they are not merely mathematical objects. They were defined as right eigenstates of a generalized annihilation operator [35, 36].

The second class of generalizations is essentially based on the overcompleteness property of coherent states. This property was the *raison d'être* of the mathematically oriented construction of generalized coherent states by Ali *et al* [3, 2] or of the ones with physical orientations [14, 15, 31]. Numerous publications continue to appear using this property, see for example [13, 22, 23, 43] and references therein. The overcompleteness property is the most important criteria to be satisfied by CS as required by Klauder's criteria [31].

To end this quick overview, let us mention the generalization performed through the so-called *coherent state map,* elaborated by Odzijewicz [44] in 1998 and generalized in [21]. It is now known that the coherent state map may be used as a tool for the geometric quantization *à la Kostant-Souriau* [44]. See the works by Kirillov [26] and Kostant [34] for details on geometric quantization.

Definition 1.3. We call deformed Heisenberg algebra, an associative algebra generated by the set of operators $\{\mathbf{1},\ a,\ a^\dagger,\ N\}$ satisfying the relations

$$[N,\ a^\dagger] = a^\dagger, \qquad [N,\ a] = -a, \tag{1.16}$$

such that there exists a non-negative analytic function φ, called the structure function, defining the operator products $a^\dagger a$ and $aa^\dagger$ in the following way:

$$a^\dagger a := \varphi(N), \qquad aa^\dagger := \varphi(N+\mathbf{1}), \tag{1.17}$$

where N is a self-adjoint operator, a and its Hermitian conjugate $a^\dagger$ denote the deformed annihilation and creation operators, respectively.

The function φ, encoding all required information, for instance, in the construction of irreducible representations of the algebra, remains the main task to solve when one deals with such deformed algebra (1.16). Different approaches for its determination are spread in the literature. See [7, 11, 12, 33, 37] and references therein. More importantly, as it will be shown in the sequel, the structure function will be the key for the unification of the coherent state construction methods from generalized algebras, respecting Klauder criteria. Note that the method put forward by Klauder [31] is based on an appropriate choice of a

set of strictly positive parameters. In the present paper, such a set of positive parameters is determined by the structure function.

The paper is organized as follows. In Section 2, the deformed Heisenberg algebra is described and the structure function is deduced. The spectrum of the associated deformed oscillator is computed. The Section 3 is devoted to the construction of the deformed coherent states using the Klauder approach. In section 4, quantum statistics and geometry of the deformed coherent states are investigated. Concluding remarks end the paper in Section 5.

2 $(q;l,\lambda)$-deformed Heisenberg algebra

Consider now the following $(q;l,\lambda)$-deformed Heisenberg algebra generated by operators N, a, $a^\dagger$ satisfying

$$[N,\, a] = -a, \qquad [N,\, a^\dagger] = a^\dagger, \tag{2.1}$$

with the operator products

$$aa^\dagger - a^\dagger a = l^2 q^{-N+\lambda-1}. \tag{2.2}$$

One can readily check that the commutator [., .] of operators is antisymmetric and satisfies the Jacobi identity conferring a Lie algebra structure to the $(q;l,\lambda)$-deformed Heisenberg algebra. This algebra plays an important role in mathematical sciences in general, and, in particular, in mathematical physics. In a notable work [25], similar associative algebra has been investigated by Kalnins *et al* under the form:

$$\begin{aligned}&[H,\, E_+] = E_+ \qquad [H,\, E_-] = -E_-\\ &[E_+,\, E_-] = -q^{-H}\mathcal{E} \qquad [\mathcal{E},\, E_\pm] = 0 = [\mathcal{E},\, H],\end{aligned} \tag{2.3}$$

where q is a real number such that $0 < q < 1$. These authors showed that the elements $C = qq^{-H}\mathcal{E} + (q-1)E_+E_-$ and $\mathcal{E}$ lie in the center of this algebra. It admits a class of irreducible representations for $C = l^2 I$ and $\mathcal{E} = l^2 q^{\lambda-1} I$, where l and λ are real numbers with $l \neq 0$.

The $(q;l,\lambda)$-deformed Heisenberg algebra (2.1) is a generalized algebra in the sense that it can generate a series of existing algebras as particular cases. For instance, even the generalization of the Quesne-algebra performed in [22, 46] can be deduced from (2.1) by setting $l = 1$ and $\lambda = 0$.

In the sequel, we consider the Fock space of the Bose oscillator constructed as follows. From the vacuum vector $|0\rangle$ defined by $a|0\rangle = 0$, the normalized vectors $|n\rangle$ for $n \geq 1$, i.e. eigenvectors of the operator N, are obtained as $|n\rangle = C_n (a^\dagger)^n |0\rangle$, where C_n stands for some normalization constant to be determined.

Proposition 2.1. *The structure function of the* $(q;l,\lambda)$*-deformed Heisenberg algebra* (2.1)–(2.2) *is given by*

$$\varphi(n) = l^2 q^\lambda \frac{1-q^{-n}}{q-1} = l^2 q^{\lambda-n}[n]_q, \;\; q > 0, \tag{2.4}$$

where $[n]_q = \frac{1-q^n}{1-q}$, *with* $0 < q < 1$ *or* $1 < q$, *is the* q_n- *factors (also known as q-deformed numbers in Physics literature [16]).*

Proof: From the definition (1.17), $a^\dagger a = \varphi(N)$ and $aa^\dagger = \varphi(N+1)$. Thus, (2.2) is written as

$$\varphi(N+\mathbf{1}) - \varphi(N) = l^2 q^{-N+\lambda-1}.$$

Applying this relation to the vectors $|n\rangle$, we obtain the recurrence relation

$$\varphi(n+1) - \varphi(n) = l^2 q^{\lambda-n-1}, \quad \forall n \in \mathbb{N}$$

from which we deduce

$$\varphi(n) = \varphi(0) + l^2 q^\lambda \frac{1-q^{-n}}{q-1}.$$

Since, in particular, $\varphi(N)|0\rangle = a^\dagger a|0\rangle = 0$ implies $\varphi(0)|0\rangle = 0$, we have $\varphi(0) = 0$. Then (2.4) follows. The structure function is also a strictly increasing function for $x \in \mathbb{R}$ since

$$\frac{d\varphi(x)}{dx} = l^2 q^{\lambda-x} \frac{\ln q}{q-1} > 0, \text{ for } q > 0.$$

Since $\varphi(0) = 0$, it follows that $\varphi(x) \geq 0$ for any real $x > 0$ and in particular $\varphi(n) \geq 0$, $\forall n \geq 0$. □

Proposition 2.2. *The orthonormalized basis of the Fock space $\mathcal{F}$ is given by*

$$|n\rangle = \frac{q^{n(n+1)/4}}{\sqrt{(l^2q^\lambda)^n [n]_q!}} (a^\dagger)^n |0\rangle, \qquad n = 0,\ 1,\ 2,\ \ldots \tag{2.5}$$

where $[0]_q! := 1$ *and* $[n]_q! := [n]_q[n-1]_q \ldots [1]_q$.
Moreover, the action of the operators a, $a^\dagger$, N, $a^\dagger a$ and $aa^\dagger$ on the vectors $|n\rangle$ for $n \geq 1$ are given by

$$a|n\rangle = \sqrt{l^2 q^{\lambda-n}[n]_q}|n-1\rangle, \tag{2.6}$$

$$a^\dagger|n\rangle = \sqrt{l^2 q^{\lambda-n-1}[n+1]_q}|n+1\rangle, \tag{2.7}$$

$$N|n\rangle = n|n\rangle, \tag{2.8}$$

$$a^\dagger a|n\rangle = l^2 q^{\lambda-n}[n]_q|n\rangle, \tag{2.9}$$

$$aa^\dagger|n\rangle = l^2 q^{\lambda-n-1}[n+1]_q|n\rangle. \tag{2.10}$$

Proof: To determine the constant of normalization C_n, we set

$$1 =: \langle n|n\rangle = |C_n|^2 \langle 0|a^n (a^\dagger)^n|0\rangle = |C_n|^2 \varphi(n)\varphi(n-1)\ldots\varphi(1)\langle 0|0\rangle$$

leading to $C_n = \frac{q^{n(n+1)/4}}{\sqrt{(l^2q^\lambda)^n[n]_q!}}$. Replacing C_n by its value in the definition of $|n\rangle$ given above yields (2.5). The orthogonality of the vectors $|n\rangle$ is a direct consequence of $a|0\rangle = 0$. The rest of the proof is obtained from (2.5) using (2.1), (2.2) and (2.4). □

Theorem 2.3. *The operators $(a + a^\dagger)$ and $i(a - a^\dagger)$, defined on the Fock space $\mathcal{F}$, are bounded and, consequently, self-adjoint if $q > 1$. If $q < 1$, they are not self-adjoint.*

Proof: The matrix elements of the operator $(a+a^\dagger)$ on the basis $|n\rangle$ are given by

$$\langle m|(a+a^\dagger)|n\rangle = x_n\delta_{m,n-1} + x_{n+1}\delta_{m,n+1}, n,\ m = 0,\ 1,\ 2,\ \cdots \tag{2.11}$$

while the matrix elements of the operator $i(a-a^\dagger)$ are given by

$$\langle m|i(a-a^\dagger)|n\rangle = ix_n\delta_{m,n-1} - ix_{n+1}\delta_{m,n+1}, n,\ m = 0,\ 1,\ 2,\ \cdots \tag{2.12}$$

where $x_n = \left(l^2 q^{\lambda-n}[n]q\right)^{1/2}$. Besides, the operators $(a+a^\dagger)$ and $i(a-a^\dagger)$ can be represented by the two following symmetric Jacobi matrices, respectively:

$$\begin{pmatrix} 0 & x_1 & 0 & 0 & 0 & \cdots \\ x_1 & 0 & x_2 & 0 & 0 & \cdots \\ 0 & x_2 & 0 & x_3 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.13}$$

and

$$\begin{pmatrix} 0 & -ix_1 & 0 & 0 & 0 & \cdots \\ ix_1 & 0 & -ix_2 & 0 & 0 & \cdots \\ 0 & ix_2 & 0 & -ix_3 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.14}$$

Two situations deserve investigation:
• Suppose $q>1$. Then,

$$|x_n| = \left(\frac{l^2q^\lambda}{q-1}\frac{q^n-1}{q^n}\right)^{1/2} < \left(\frac{l^2q^\lambda}{q-1}\right)^{1/2},\ \forall n \geq 1.$$

Therefore, the Jacobi matrices in (2.13) and (2.14) are bounded and self-adjoint (Theorem 1.2., Chapter VII in Ref. [9]). Thus, $(a+a^\dagger)$ and $i(a-a^\dagger)$ are bounded and, consequently, self-adjoint.
• Contrarily, if $q<1$, then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(l^2q^\lambda\frac{1-q^{-n}}{q-1}\right)^{1/2} = \infty. \tag{2.15}$$

Considering the series $\sum_{n=1}^{\infty} 1/x_n$, we obtain

$$\overline{\lim_{n\to\infty}}\left(\frac{1/x_{n+1}}{1/x_n}\right) = \overline{\lim_{n\to\infty}}\left(\frac{1-q^{-n}}{1-q^{-n-1}}\right)^{1/2} = q^{1/2} < 1.$$

This ratio test leads to the conclusion that the series $\sum_{n=1}^{\infty} 1/x_n$ converges. Moreover, $1-2q+q^2 = (1-q)^2 \geq 0 \Longrightarrow q^{-1}+q \geq 2$. Hence,

$$0 \leq \left(\frac{l^2q^\lambda}{q-1}\right)^2\left(1-q^{-n}(q+q^{-1})+q^{-2n}\right) \leq \left(1-2q^{-n}+q^{-2n}\right)\left(\frac{l^2q^\lambda}{q-1}\right)^2$$

$$\begin{array}{ll}\Leftrightarrow & 0 \le \left(l^2 q^\lambda \frac{1-q^{-n+1}}{q-1}\right)\left(l^2 q^\lambda \frac{1-q^{-n-1}}{q-1}\right) \le \left(l^2 q^\lambda \frac{1-q^{-n}}{q-1}\right)^2 \\ \Leftrightarrow & 0 \le \left(l^2 q^\lambda \frac{1-q^{-n+1}}{q-1}\right)^{1/2}\left(l^2 q^\lambda \frac{1-q^{-n-1}}{q-1}\right)^{1/2} \le \left(l^2 q^\lambda \frac{1-q^{-n}}{q-1}\right) \\ \Leftrightarrow & 0 \le x_{n-1} x_{n+1} \le x_n^2.\end{array}$$

Therefore, the Jacobi matrices in (2.13) and (2.14) are not self-adjoint (Theorem 1.5., Chapter VII in Ref. [9]). □

Definition 2.4. The $(q;l,\lambda)$-deformed position, momentum and Hamiltonian operators denoted by $X_{l,\lambda,q}$, $P_{l,\lambda,q}$ and $H_{l,\lambda,q}$, respectively, are defined as follows:

$$\begin{aligned} X_{l,\lambda,q} &:= (\hbar/2\mathbf{m}\omega)^{1/2}\,(a+a^\dagger), \\ P_{l,\lambda,q} &:= -i\,(\mathbf{m}\hbar\omega/2)^{1/2}\,(a-a^\dagger) \\ H_{l,\lambda,q} &:= \frac{1}{2\mathbf{m}}(P_{l,\lambda,q})^2 + \frac{1}{2}\mathbf{m}\omega^2 (X_{l,\lambda,q})^2 \\ &= \frac{\hbar\omega}{2}(a^\dagger a + a a^\dagger). \end{aligned} \tag{2.16}$$

Proposition 2.5. *The following system characterization holds:*

- *The vectors $|n\rangle$ are eigenvectors of the $(q;l,\lambda)$-deformed Hamiltonian with respect to the eigenvalues*

$$E_{l,\lambda,q}(n) = \frac{\hbar\omega}{2} l^2 q^{\lambda-n-1}(q[n]_q + [n+1]_q). \tag{2.17}$$

- *The mean values of $X_{l,\lambda,q}$ and $P_{l,\lambda,q}$ in the states $|n\rangle$ are zero while their variances are given by*

$$(\Delta X_{l,\lambda,q})_n^2 = \frac{\mathbf{m}\hbar\omega}{2} l^2 q^{\lambda-n-1}(q[n]_q + [n+1]_q), \tag{2.18}$$

$$(\Delta P_{l,\lambda,q})_n^2 = \frac{\hbar}{2\mathbf{m}\omega} l^2 q^{\lambda-n-1}(q[n]_q + [n+1]_q), \tag{2.19}$$

where $(\Delta A)_n^2 = \langle A^2\rangle_n - \langle A\rangle_n^2$ with $\langle A\rangle_n = \langle n|A|n\rangle$.

- *The position-momentum uncertainty relation is given by*

$$(\Delta X_{l,\lambda,q})_n (\Delta P_{l,\lambda,q})_n = \frac{h}{2} l^2 q^{\lambda-n-1}(q[n]_q + [n+1]_q) \tag{2.20}$$

which is reduced, for the vacuum state, to the expression

$$(\Delta X_{l,\lambda,q})_0 (\Delta P_{l,\lambda,q})_0 = \frac{h}{2} l^2 q^{\lambda-1}. \tag{2.21}$$

Proof: Indeed, using the result of the Proposition 2.2, we get

$$H_{l,\lambda,q}|n\rangle = \frac{\hbar\omega}{2}(a^\dagger a + a a^\dagger))\rangle = \frac{\hbar\omega}{2} l^2 q^{\lambda-n-1}(q[n]_q + [n+1]_q)|n\rangle.$$

The first two relations (2.11) and (2.12) in the proof of the previous Theorem 2.3 yield $\langle n|(a+a^\dagger)|n\rangle = 0 = \langle n|i(a-a^\dagger)|n\rangle$ and $\langle n|(a+a^\dagger)^2|n\rangle = x_n^2 + x_{n+1}^2 = \langle n|i^2(a-a^\dagger)^2|n\rangle$. Therefore, $\langle n|X_{l,\lambda}|n\rangle = 0 = \langle n|P_{l,\lambda}|n\rangle$, $\langle n|X_{l,\lambda}^2|n\rangle = \frac{\mathbf{m}\hbar\omega}{2}(x_n^2 + x_{n+1}^2)$ and $\langle n|P_{l,\lambda}^2|n\rangle = \frac{\hbar}{2\mathbf{m}\omega}(x_n^2 + x_{n+1}^2)$. The rest of the proof is obtained replacing x_n and x_{n+1} by their expressions. □

3 Coherent states $|z\rangle_{l,\lambda}$

Definition 3.1. The coherent states associated with the algebra (2.1)-(2.2) are defined as

$$|z\rangle_{l,\lambda} := \mathcal{N}_{l,\lambda}^{-1/2}(|z|^2)\sum_{n=0}^{\infty}\frac{q^{n(n+1)/4}z^n}{\sqrt{(l^2q^\lambda)^n[n]_q!}}|n\rangle,\ z\in\mathbf{D}_{l,\lambda}, \tag{3.1}$$

where

$$\mathcal{N}_{l,\lambda}(x) \quad = \quad \sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}x^n}{(l^2q^\lambda)^n[n]_q!} = \sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}}{(q;q)_n}\left(\frac{(1-q)qx}{l^2q^\lambda}\right)^n \tag{3.2}$$

and

$$\mathbf{D}_{l,\lambda} = \left\{z\in\mathbb{C}:\ |z|^2<R_{l,\lambda}\right\},\ \text{with}\ \ R_{l,\lambda} = \begin{cases}\infty & \text{if}\quad 0<q<1\\ \frac{l^2q^\lambda}{q-1} & \text{if}\quad q>1.\end{cases} \tag{3.3}$$

$R_{l,\lambda}$ is the convergence radius of the series $\mathcal{N}_{l,\lambda}(x)$.

Remark that the q-deformed coherent states introduced in [46] are recovered as a particular case corresponding to $l=1$ and $\lambda=0$.

We now aim at showing that the coherent states (3.1) satisfy the Klauder's criteria [30, 31]. To this end let us first prove the following lemma:

Lemma 3.2. *If* $q>1$, *then*

$$\frac{\mathcal{N}_{l,\lambda}(x)}{\mathcal{N}_{l,\lambda}(q^{-1}x)} = \frac{1}{1-(q-1)x/(l^2q^\lambda)}\ , \tag{3.4}$$

$$\mathcal{N}_{l,\lambda}(x) = \frac{1}{((q-1)x/(l^2q^\lambda);q^{-1})_\infty}, \tag{3.5}$$

$$\int_0^{R_{l,\lambda}} x^n\left(\mathcal{N}_{l,\lambda}(q^{-1}x)\right)^{-1}d_q^{l,\lambda}x = (l^2q^\lambda)^n q^{-n(n+1)/2}[n]_q!. \tag{3.6}$$

Proof: We use the $(q;l,\lambda)$-derivative defined by

$$\partial_q^{l,\lambda}f(x) = l^2q^\lambda\frac{f(x)-f(q^{-1}x)}{(q-1)x} \tag{3.7}$$

to obtain

$$\mathcal{N}_{l,\lambda}(x) = \partial_q^{l,\lambda}\mathcal{N}_{l,\lambda}(x) = l^2q^\lambda\frac{\mathcal{N}_{l,\lambda}(x)-\mathcal{N}_{l,\lambda}(q^{-1}x)}{(q-1)x}$$

which leads to (3.4) and

$$\mathcal{N}_{l,\lambda}(x) = \frac{\mathcal{N}_{l,\lambda}(q^{-n}x)}{\prod_{k=0}^{n-1}(1-(q-1)q^{-k}x/(l^2q^\lambda))},\ n=1,\ 2,\ ... \tag{3.8}$$

Letting n to $+\infty$ and taking into account the fact that $\mathcal{N}_{l,\lambda}(0)=1$ lead to (3.5).

Next, we use the $(q;l,\lambda)$-integration given by

$$\int_0^a f(x)d_q^{l,\lambda}x = \frac{q-1}{l^2q^\lambda}a\sum_{k=0}^{\infty}q^{-k}f(aq^{-k}) \tag{3.9}$$

to get

$$\begin{aligned}\int_0^{R_{l,\lambda}} x^n\left(\mathcal{N}_{l,\lambda}(q^{-1}x)\right)^{-1}d_q^{l,\lambda}x &= \sum_{k=0}^{\infty}q^{-(n+1)k}\frac{(l^2q^\lambda)^n}{(q-1)^n}(q^{-(k+1)};q^{-1})_\infty \\ &= \frac{(l^2q^\lambda)^n}{(q-1)^n}(q^{-1};q^{-1})_\infty\sum_{k=0}^{\infty}\frac{q^{-(n+1)k}}{(q^{-1};q^{-1})_k} \\ &= \frac{(l^2q^\lambda)^n}{(q-1)^n}\frac{(q^{-1};q^{-1})_\infty}{(q^{-(n+1)};q^{-1})_\infty} \\ &= \frac{(l^2q^\lambda)^n}{(q-1)^n}(q^{-1};q^{-1})_n = (l^2q^\lambda)^nq^{-n(n+1)/2}[n]_q!.\end{aligned}$$

□

Proposition 3.3. *The coherent states defined in (3.1)*

(i) are normalized eigenvectors of the operator a with eigenvalue z, i.e.

$$a|z\rangle_{l,\lambda} = z|z\rangle_{l,\lambda}, \qquad {}_{l,\lambda}\langle z|z\rangle_{l,\lambda} = 1; \tag{3.10}$$

(ii) are not orthogonal to each other, i.e.

$${}_{l,\lambda}\langle z_1|z_2\rangle_{l,\lambda} \neq 0, \text{ when } z_1 \neq z_2; \tag{3.11}$$

(iii) are continuous in their labels z;

(iv) resolve the unity, i.e.

$$\mathbf{1} = \int_{\mathbf{D}_{l,\lambda}} d\mu_{l,\lambda}(\bar{z},z)|z\rangle_{l,\lambda\,l,\lambda}\langle z|, \tag{3.12}$$

where

$$d\mu_{l,\lambda}(\bar{z},z) = \frac{1-q}{l^2q^\lambda\ln q^{-1}}\frac{\mathcal{N}_{l,\lambda}(\bar{z}z)}{\mathcal{N}_{l,\lambda}(\bar{z}z/q)}\frac{d^2z}{\pi}, \text{ if } 0<q<1, \tag{3.13}$$

and

$$d\mu(\bar{z},z) = \frac{1}{2\pi}\frac{d_q^{l,\lambda}x\,d\theta}{1-(q-1)x/(l^2q^\lambda)}, \quad x=|z|^2, \ \theta=\arg(z), \tag{3.14}$$

with $0<x<\frac{l^2q^\lambda}{q-1}$ *and* $0\le\theta\le 2\pi$ *for* $q>1$.

Proof:

•*Non orthogonality and normalizability*

$$ {}_{l,\lambda}\langle z_1|z_2\rangle_{l,\lambda} = \frac{\mathcal{N}_{l,\lambda}(\bar{z}_1 z_2)}{\left(\mathcal{N}_{l,\lambda}(|z_1|^2)\mathcal{N}_{l,\lambda}(|z_2|^2)\right)^{1/2}} \neq 0 \tag{3.15} $$

imply that the coherent states are not orthogonal.

•*Normalizability*

From the above relation taking $z_1 = z_2 = z$ we obtain ${}_{l,\lambda}\langle z|z\rangle_{l,\lambda} = 1$. Also,

$$ \begin{aligned} a|z\rangle_{l,\lambda} &= \mathcal{N}_{l,\lambda}^{-1/2}(|z|^2)\sum_{n=0}^{\infty}\frac{q^{n(n+1)/4}z^n}{\sqrt{(l^2q^\lambda)^n[n]_q!}}a|n\rangle \\ &= \mathcal{N}_{l,\lambda}^{-1/2}(|z|^2)\sum_{n=1}^{\infty}\frac{q^{n(n-1)/4}z^n}{\sqrt{(l^2q^\lambda)^{n-1}[n-1]_q!}}|n-1\rangle \\ &= z\mathcal{N}_{l,\lambda}^{-1/2}(|z|^2)\sum_{n=0}^{\infty}\frac{q^{n(n+1)/4}z^n}{\sqrt{(l^2q^\lambda)^n[n]_q!}}|n\rangle. \end{aligned} $$

•*Continuity in the labels z*

$$ \||z_1\rangle_{l,\lambda} - |z_2\rangle_{l,\lambda}\|^2 = 2\left(1 - \mathcal{R}e_{l,\lambda}\langle z_1|z_2\rangle_{l,\lambda}\right). $$

So, $\||z_1\rangle_{l,\lambda} - |z_2\rangle_{l,\lambda}\|^2 \to 0$ as $|z_1 - z_2| \to 0$, since ${}_{l,\lambda}\langle z_1|z_2\rangle_{l,\lambda} \to 1$ as $|z_1 - z_2| \to 0$.

•*Resolution of the unity*

The computation of the RHS of (3.12) gives

$$ \int_{\mathbf{D}_{l,\lambda}} d\mu_{l,\lambda}(\bar{z},z)|z\rangle_{l,\lambda\,l,\lambda}\langle z| = \sum_{n,m}|n\rangle\langle m|\frac{q^{[n(n+1)+m(m+1)]/4}}{\sqrt{(l^2q^\lambda)^{n+m}[n]_q![m]_q!}}\int_{\mathbf{D}_{l,\lambda}}\bar{z}^n z^m\frac{d\mu_{l,\lambda}(\bar{z},z)}{\mathcal{N}_{l,\lambda}(|z|^2)}. \tag{3.16} $$

So, in order to satisfy (3.12) it is required

$$ \int_{\mathbf{D}_{l,\lambda}}\bar{z}^n z^m\frac{d\mu_{l,\lambda}(\bar{z},z)}{\mathcal{N}_{l,\lambda}(|z|^2)} = \delta_{mn}(l^2q^\lambda)^n q^{-n(n+1)/2}[n]_q!, \quad n,\ m = 0,\ 1,\ 2,\ ... \tag{3.17} $$

Upon passing to polar coordinates, $z = \sqrt{x}\,e^{i\theta}$, $d\mu_{l,\lambda}(\bar{z},z) = d\omega_{l,\lambda}(x)d\theta$ where $0 \le \theta \le 2\pi$, $0 < x < R_{l,\lambda}$ and $\omega_{l,\lambda}$ is a positive valued function, this is equivalent to the classical Stieltjes power moment problem when $0 < q < 1$ or the Hausdorff power moment problem when $q > 1$ [1, 49]:

$$ \int_0^{R_{l,\lambda}} x^n\,\frac{2\pi\,d\omega_{l,\lambda}(x)}{\mathcal{N}_{l,\lambda}(x)} = (l^2q^\lambda)^n q^{-n(n+1)/2}[n]_q!, \quad n = 0,\ 1,\ 2,\ ... \tag{3.18} $$

If $0 < q < 1$, then we have the following Stieltjes power moment problem:

$$ \int_0^{+\infty} x^n\frac{2\pi\,d\omega_{l,\lambda}(x)}{\mathcal{N}_{l,\lambda}(x)} = (l^2q^\lambda)^n q^{-n(n+1)/2}[n]_q!, \tag{3.19} $$

or, equivalently,

$$\int_0^{+\infty} y^n \frac{2\pi\, d\omega_{l,\lambda}(l^2q^\lambda y)}{E_q((1-q)qy)} = q^{-n(n+1)/2}[n]_q!, \tag{3.20}$$

where the change of variable $y = \frac{x}{l^2q^\lambda}$ has been made. Atakishiyev and Atakishiyeva [6] have proved that

$$g_q(n) = \int_0^{+\infty} \frac{y^{n-1}dy}{E_q((1-q)y)} = \frac{\ln q^{-1}}{1-q} q^{-n(n-1)/2}[n-1]!_q. \tag{3.21}$$

Therefore we deduce

$$d\omega_{l,\lambda}(l^2q^\lambda y) = \frac{1}{2\pi}\frac{1-q}{\ln q^{-1}}\frac{E_q((1-q)qy)dy}{E_q((1-q)y)}$$

or

$$\begin{aligned} d\omega_{l,\lambda}(x) &= \frac{1}{2\pi}\frac{1-q}{l^2q^\lambda \ln q^{-1}}\frac{E_q((1-q)qx/(l^2q^\lambda))dx}{E_q((1-q)x/(l^2q^\lambda))} \\ &= \frac{1}{2\pi}\frac{1-q}{l^2q^\lambda \ln q^{-1}}\frac{\mathcal{N}_{l,\lambda}(x)dx}{\mathcal{N}_{l,\lambda}(x/q)}. \end{aligned} \tag{3.22}$$

Hence

$$d\mu_{l,\lambda}(\bar{z},z) = \frac{1-q}{l^2q^\lambda \ln q^{-1}}\frac{\mathcal{N}_{l,\lambda}(\bar{z}z)}{\mathcal{N}_{l,\lambda}(\bar{z}z/q)}\frac{d^2z}{\pi}. \tag{3.23}$$

In the opposite, if $q > 1$, then combining (3.18), (3.4) and (3.5) of the Lemma 3.2 we get

$$d\mu(\bar{z},z) = \frac{1}{2\pi}\frac{d_q^{l,\lambda} x\, d\theta}{1-(q-1)x/(l^2q^\lambda)}, \quad x = |z|^2, \ \theta = \arg(z), \tag{3.24}$$

where $0 < x < \frac{l^2q^\lambda}{q-1}$ and $0 \le \theta \le 2\pi$. □

4 Statistics and geometry of coherent states $|z\rangle_{l,\lambda}$

The conventional boson operators b and $b^\dagger$ may be expressed in terms of the deformed operators a and $a^\dagger$ as

$$b = a\sqrt{\frac{N}{\varphi(N)}} \quad \text{and} \quad b^\dagger = \sqrt{\frac{N}{\varphi(N)}}\, a^\dagger, \quad \varphi(N) \neq \varphi(0) \tag{4.1}$$

and their actions on the states $|n\rangle$ are given by

$$b|n\rangle = \sqrt{n}|n-1\rangle, \quad \text{and} \quad b^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{4.2}$$

Besides,

$$b^r|n\rangle = \sqrt{\frac{n!}{(n-r)!}}|n-r\rangle, \qquad 0 \le r \le n \tag{4.3}$$

and

$$(b^\dagger)^s|n\rangle = \sqrt{\frac{(n+s)!}{n!}}|n+s\rangle. \tag{4.4}$$

4.1 *Quantum statistics of the coherent states $|z\rangle_{l,\lambda}$*

Proposition 4.1. *The expectation value of monomials of boson creation and annihilation operators $b^\dagger$, b in the coherent states $|z\rangle_{l,\lambda}$ are given by*

$$\langle (b^\dagger)^s b^r\rangle = \frac{\bar{z}^s z^r}{\mathcal{N}_{l,\lambda}(|z|^2)}\sum_{n=0}^{\infty}\sqrt{\frac{q^{[(n+s)(n+s+1)+(n+r)(n+r+1)]/2}(n+r)!(n+s)!}{(l^2q^\lambda)^{(n+s)+(n+r)}[n+s]_q![n+r]_q!}}\frac{|z|^{2n}}{n!}, \tag{4.5}$$

where $s = 0,\ 1,\ 2,\cdots$ and $r = 0,\ 1,\ 2,\cdots$.
In particular,

$$\langle (b^\dagger)^r b^r\rangle = \frac{x^r}{\mathcal{N}_{l,\lambda}(x)}\left(\frac{d}{dx}\right)^r\mathcal{N}_{l,\lambda}(x), \quad x = |z|^2, \quad r = 0,\ 1,\ 2,\cdots, \tag{4.6}$$

and

$$\langle N\rangle = x\frac{\mathcal{N}'_{l,\lambda}(x)}{\mathcal{N}_{l,\lambda}(x)}, \tag{4.7}$$

where $\mathcal{N}'_{l,\lambda}(x)$ denotes the derivative with respect to x.

Proof: Indeed, for $s = 0,\ 1,\ 2,\cdots$ and $r = 0,\ 1,\ 2,\cdots$, we have

$$\begin{aligned}
\langle (b^\dagger)^s b^r\rangle &:= {}_{l,\lambda}\langle z|(b^\dagger)^s b^r|z\rangle_{l,\lambda}\\
&= \frac{1}{\mathcal{N}_{l,\lambda}(|z|^2)}\sum_{m=0}^{\infty}\sum_{n=r}^{\infty}\sqrt{\frac{q^{[m(m+1)+n(n+1)]/2}n!(n-r+s)!}{(l^2q^\lambda)^{m+n}[m]_q![n]_q!(n-r)!(n-r)!}}\bar{z}^m z^n\langle m|n+s-r\rangle\\
&= \frac{1}{\mathcal{N}_{l,\lambda}(|z|^2)}\sum_{n=r}^{\infty}\sqrt{\frac{q^{[(n+s-r)(n+s-r+1)+n(n+1)]/2}n!(n-r+s)!}{(l^2q^\lambda)^{n+s-r+n}[n+s-r]_q![n]_q!(n-r)!(n-r)!}}\bar{z}^{n+s-r}z^n\\
&= \frac{\bar{z}^s z^r}{\mathcal{N}_{l,\lambda}(|z|^2)}\sum_{n=0}^{\infty}\sqrt{\frac{q^{[(n+s)(n+s+1)+(n+r)(n+r+1)]/2}(n+r)!(n+s)!}{(l^2q^\lambda)^{(n+s)+(n+r)}[n+s]_q![n+r]_q!}}\frac{|z|^{2n}}{n!},
\end{aligned}$$

In the special case $s = r$, we have

$$\begin{aligned}
\langle (b^\dagger)^r b^r\rangle &= \frac{x^r}{\mathcal{N}_{l,\lambda}(x)}\sum_{n=0}^{\infty}\frac{q^{(n+r)(n+r+1)/2}(n+r)!}{(l^2q^\lambda)^{(n+r)}[n+r]_q!}\frac{x^n}{n!}\\
&= \frac{x^r}{\mathcal{N}_{l,\lambda}(x)}\sum_{n=r}^{\infty}\frac{q^{n(n+1)/2}(n)!}{(l^2q^\lambda)^{(n)}[n]_q!}\frac{x^{n-r}}{(n-r)!}\\
&= \frac{x^r}{\mathcal{N}_{l,\lambda}(x)}\left(\frac{d}{dx}\right)^r\mathcal{N}_{l,\lambda}(x), \quad x = |z|^2.
\end{aligned}$$

In particular

$$\langle N\rangle \equiv \langle b^\dagger b\rangle = x\frac{\mathcal{N}'_{l,\lambda}(x)}{\mathcal{N}_{l,\lambda}(x)}.$$

□

The probability of finding n quanta in the deformed state $|z\rangle_{l,\lambda}$ is given by

$$\mathcal{P}_{l,\lambda}(n) := |\langle n|z\rangle_{l,\lambda}|^2 = \frac{q^{n(n+1)/2} x^n}{(l^2 q^\lambda)^n [n]_q! \mathcal{N}_{l,\lambda}(x)}. \tag{4.8}$$

The Mendel parameter measuring the deviation from the Poisson statistics is defined by the quantity

$$Q_{l,\lambda} := \frac{\langle N^2\rangle - \langle N\rangle^2 - \langle N\rangle}{\langle N\rangle}. \tag{4.9}$$

Let us evaluate it explicitly. From the expectation value of the operator $N^2 = (b^\dagger)^2 b^2 + N$ provided by

$$\langle N^2\rangle = x^2 \frac{\mathcal{N}''_{l,\lambda}(x)}{\mathcal{N}_{l,\lambda}(x)} + x\frac{\mathcal{N}'_{l,\lambda}(x)}{\mathcal{N}_{l,\lambda}(x)}, \tag{4.10}$$

we readily deduce

$$Q_{l,\lambda} = x\left(\frac{\mathcal{N}''_{l,\lambda}(x)}{\mathcal{N}'_{l,\lambda}(x)} - \frac{\mathcal{N}'_{l,\lambda}(x)}{\mathcal{N}_{l,\lambda}(x)}\right). \tag{4.11}$$

It is then worth noticing that for $x << 1$,

$$Q_{l,\lambda} = -\frac{q(1-q)}{l^2 q^\lambda (1+q)} x + o(x^2) \tag{4.12}$$

meaning that the $\mathcal{P}_{l,\lambda}(n)$ is a sub-Poissonian distribution [31].

4.2 *Geometry of the states $|z\rangle_{l,\lambda}$*

The geometry of a quantum state space can be described by the corresponding metric tensor. This real and positive definite metric is defined on the underlying manifold that the quantum states form, or belong to, by calculating the distance function (line element) between two quantum states. So, it is also known as a Fubini-Study metric of the ray space. The knowledge of the quantum metric enables one to calculate quantum mechanical transition probability and uncertainties

In the case $q < 1$, the map from z to $|z\rangle_{l,\lambda}$ defines a map from the space $\mathbb{C}$ of complex numbers onto a continuous subset of unit vectors in Hilbert space and generates in the latter a two-dimensional surface with the following Fubini-Study metric:

$$d\sigma^2 := ||d|z\rangle_{l,\lambda}||^2 - |_{l,\lambda}\langle z|d|z\rangle_{l,\lambda}|^2 \tag{4.13}$$

Proposition 4.2. *The above Fubini-Study metric is reduced to*

$$d\sigma^2 = W_{l,\lambda}(x) d\bar{z} dz, \tag{4.14}$$

where $x = |z|^2$ and

$$W_{l,\lambda}(x) = \left(x\frac{\mathcal{N}'_{l,\lambda}(x)}{\mathcal{N}_{l,\lambda}(x)}\right)' = \frac{d}{dx}\langle N\rangle. \tag{4.15}$$

In polar coordinates, $z = re^{i\theta}$,

$$d\sigma^2 = W_{l,\lambda}(r^2)(dr^2 + r^2 d\theta^2). \tag{4.16}$$

Proof: Computing $d|z\rangle_{l,\lambda}$ by taking into account the fact that any change of the form $d|z\rangle_{l,\lambda} = \alpha|z\rangle_{l,\lambda}$, $\alpha \in \mathbb{C}$, has zero distance, we get

$$d|z\rangle_{l,\lambda} = \mathcal{N}_{l,\lambda}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/4} n z^{n-1}}{\sqrt{(l^2 q^\lambda)^n [n]_q!}} |n\rangle \, dz.$$

Then,

$$\begin{aligned}
\||d|z\rangle_{l,\lambda}\|^2 &= \mathcal{N}_{l,\lambda}(|z|^2)^{-1} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} n^2 |z|^{2(n-1)}}{(l^2 q^\lambda)^n [n]_q!} d\bar{z}dz \\
&= \mathcal{N}_{l,\lambda}(|z|^2)^{-1} \left(\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} n |z|^{2(n-1)}}{(l^2 q^\lambda)^n [n]_q!} \right. \\
&\qquad \left. + |z|^2 \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} n(n-1) |z|^{2(n-2)}}{(l^2 q^\lambda)^n [n]_q!} \right) d\bar{z}dz \\
&= \mathcal{N}_{l,\lambda}(x)^{-1} \left(\mathcal{N}'_{l,\lambda}(x) + x \mathcal{N}''_{l,\lambda}(x) \right) d\bar{z}dz \\
&= \mathcal{N}_{l,\lambda}(x)^{-1} \left(x \mathcal{N}'_{l,\lambda}(x) \right)' d\bar{z}dz
\end{aligned}$$

and

$$\begin{aligned}
|{}_{l,\lambda}\langle z|d|z\rangle_{l,\lambda}|^2 &= \left| \mathcal{N}_{l,\lambda}(|z|^2)^{-1} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2} n |z|^{2(n-1)}}{(l^2 q^\lambda)^n [n]_q!} \bar{z} dz \right|^2 \\
&= x \mathcal{N}_{l,\lambda}(x)^{-2} \left(\mathcal{N}'_{l,\lambda}(x) \right)^2 d\bar{z}dz.
\end{aligned}$$

Therefore,

$$\begin{aligned}
d\sigma^2 &= \left(\mathcal{N}_{l,\lambda}(x)^{-1} \left(\mathcal{N}'_{l,\lambda}(x) + x \mathcal{N}''_{l,\lambda}(x) \right) - x \mathcal{N}_{l,\lambda}(x)^{-2} \left(\mathcal{N}'_{l,\lambda}(x) \right)^2 \right) d\bar{z}dz \\
&= \left(x \frac{\mathcal{N}'_{l,\lambda}(x)}{\mathcal{N}_{l,\lambda}(x)} \right)' d\bar{z}dz = \left(\frac{d}{dx} \langle N \rangle \right) d\bar{z}dz.
\end{aligned}$$

□

For $x << 1$, we have

$$W_{l,\lambda}(x) = \frac{q}{l^2 q^\lambda} \left[1 - \frac{2q(1-q)}{l^2 q^\lambda (1+q)} x + o(x^2) \right]. \quad (4.17)$$

5 Concluding remark

In the present work, we have deformed the Heisenberg algebra with the set of parameters $\{q, l, \lambda\}$ to generate a new family of generalized coherent states respecting the Klauder criteria. In this framework, the matrix elements of relevant operators have been exactly computed and investigated from functional analysis point of view. Then, relevant statistical properties have been examined. Besides, a proof on the sub-Poissonian character of the statistics of the main deformed states has been provided. This property has been finally used to determine the induced generalized metric, characterizing the geometry of the considered system.

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